

On the uniqueness of stratifications of derived module categories

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Abstract. Recollements of triangulated categories may be seen as exact sequences of such categories. Iterated recollements of triangulated categories are analogues of geometric or topological stratifications and of composition series of algebraic objects. We discuss the question of uniqueness of such a stratification, up to ordering and derived equivalence, for derived module categories. The main result is a positive answer in the form of a Jordan Hölder theorem for derived module categories of hereditary artin algebras. We also provide examples of derived simple rings.

Keywords: recollement; stratification; derived simplicity; Jordan Hölder theorem; hereditary artin algebras.

1. INTRODUCTION

Classical Jordan Hölder theorems in group theory or in representation theory assert that under some finiteness assumptions a given group, module or representation has a finite composition series with simple factors and that the simple factors are unique up to ordering and isomorphism. This article proves a new kind of Jordan Hölder theorem, for derived categories of hereditary artin algebras, that is for certain triangulated categories.

The ingredients of a Jordan Hölder theorem are the terms *composition series* and *simple* objects. A finite composition series is a succession of short exact sequences. A simple object is not allowed to be the middle term of any non-trivial short exact sequence. We propose to view recollements of triangulated categories as analogues of short exact sequences. Hence iterated recollements, frequently called *stratifications*, are analogues of composition series. It then may seem natural to call a triangulated category simple if it does not admit a non-trivial recollement by triangulated categories. We will see, however, that the choice of definition is a subtle point - like in geometry it is crucial to decide which kind of subobjects or factor objects are to occur in stratifications. For derived categories of rings it turns out to be reasonable to call a derived category simple if it does not admit a non-trivial recollement, whose factors again are derived categories of rings. In this sense, the main result of this article is:

Main Theorem 5.1. *The (unbounded) derived module category of a hereditary artin algebra admits a finite composition series, and the simple factors in a composition series are unique up to ordering and equivalence of triangulated categories.*

The recollements of triangulated categories used here as analogues of exact sequences describe the middle term by a triangulated subcategory and a triangulated quotient category. Recollements have been first defined by Beilinson, Bernstein and Deligne [4] in geometric contexts, where stratifications of spaces imply recollements of derived categories of sheaves, by using derived versions of Grothendieck's six functors - whose abstract properties in fact get axiomatized by the notion of recollement. As certain derived categories of perverse sheaves are equivalent to derived categories of modules over blocks of the Bernstein-Gelfand-Gelfand category \mathcal{O} , recollements do exist for the corresponding algebras as well. For these algebras, the stratification provided by iterated recollements, is by derived categories of vector spaces. This is one of the fundamental, and motivating, properties of quasi-hereditary algebras, introduced by Cline, Parshall and Scott

[6]. Another source of examples for stratifications of algebras in this sense is a result of Beilinson [3], which identifies certain derived categories of sheaves with derived categories of algebras; this relates, for instance the coherent sheaves on a projective line with the path algebra of the Kronecker quiver. Recently, recollements of derived categories also have come up in tilting theory [1], in the context of tilting modules associated with injective homological epimorphisms.

The question of Jordan Hölder theorems being valid for (certain) derived categories or more generally uniqueness of stratifications to hold true for (certain) triangulated categories has come up about twenty years ago with the work of Cline, Parshall and Scott [6, 25]. It has motivated studies of examples by Wiedemann [28] and Happel [12] as well as the criterion for existence of recollements in [17]. Despite this interest, the main Theorem is the first positive result obtained so far for any class of algebras and of derived categories. A similar result cannot hold true for all algebras: In general, both existence and uniqueness of a finite Jordan Hölder series of a derived category may fail. A counterexample to existence will be provided in Section 6. The difficult problem of uniqueness has been taken up by Chen and Xi [5], who have constructed an algebra, whose derived category has two different Jordan Hölder series of different lengths. They also provided examples of finite Jordan Hölder series of the same derived category having the same length, but different composition factors.

Like exact sequences in abelian categories, recollements of triangulated categories have - in general or under additional assumptions - associated long exact sequences for various cohomology theories, such as K-theory, cyclic homology, and Hochschild cohomology. Moreover, the middle term and the outer terms of a recollement of derived categories of rings share some homological invariants; for instance, the middle term has finite global or finitistic dimension if and only if the outer terms have so as well, see [13].

This article is organised as follows. In Section 2 we recall the definitions needed and then we collect for later use a variety of results from various backgrounds. In Section 3 we prove technical results on completing recollement diagrams, which in our setup play the role of the butterfly lemma used in proofs of the classical Jordan Hölder theorems. In Section 4 we discuss lifting and restricting of recollements between bounded or unbounded derived categories, and we give some criteria and examples of derived simple rings. Section 5 contains the proof of the main theorem 5.1. Section 6 provides various (counter)examples; this illustrates in particular our choice of definition for 'derived simplicity'.

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2. RECOLLEMENTS, UNIVERSAL EXTENSIONS AND PERPENDICULAR CATEGORIES

In this section we recall the definition of recollements and then collect information on connections with other concepts, which will be used in the proof of the Main Theorem 5.1.

Throughout this article, rings or algebras are assumed to be associative with a unit element. An artin algebra by definition is an artinian algebra over a commutative artinian ring.

2.1. Recollements. Let \mathcal{X}, \mathcal{Y} and \mathcal{D} be triangulated categories. \mathcal{D} is said to be a *recollement* ([4], see also [25]) of \mathcal{X} and \mathcal{Y} if there are six triangle functors as in the following diagram

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{Y} & \xrightleftharpoons[i_* = i_!]{\quad} & \mathcal{D} & \xrightleftharpoons[j^! = j^*]{\quad} & \mathcal{X} \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

such that

- (1) (i^*, i_*) , $(i_!, i^!)$, $(j_!, j^!)$, (j^*, j_*) are adjoint pairs;
- (2) i_* , j_* , $j_!$ are full embeddings;
- (3) $i^! \circ j_* = 0$ (and thus also $j^! \circ i_! = 0$ and $i^* \circ j_! = 0$);
- (4) for each $C \in \mathcal{D}$ there are triangles

$$\begin{aligned} i_! i^!(C) \rightarrow C \rightarrow j_* j^*(C) &\rightsquigarrow \\ j_! j^!(C) \rightarrow C \rightarrow i_* i^*(C) &\rightsquigarrow \end{aligned}$$

where the four morphisms starting from/ending at C are the unit/counits of the adjoint pairs in (1).

In this paper, \mathcal{D} will always be a derived module category $\mathcal{D}(R) = \mathcal{D}(\text{Mod-}R)$ of some ring R (with unit). By $\text{Mod-}R$ we mean the category of right R -modules. Later on we will work with hereditary artin algebras.

2.2. Homological epimorphisms and universal localization. Let $\lambda: R \rightarrow S$ be a ring epimorphism, that is, an epimorphism in the category of rings. Following Geigle and Lenzing [11], we say that λ is a *homological ring epimorphism* if $\text{Tor}_i^R(S, S) = 0$ for all $i > 0$. Note that this holds true if and only if the restriction functor $\lambda_*: \mathcal{D}(S) \rightarrow \mathcal{D}(R)$ induced by λ is fully faithful ([11, 4.4], [22, 5.3.1]). The following result connects homological epimorphisms and recollements, where their epiclasses and equivalence classes are defined naturally. For more details see [1].

Proposition 2.1. ([1], 1.7) *There is a bijection between the epiclasses of homological ring epimorphisms and the equivalence classes of those recollements*

$$\mathcal{Y} \quad \xrightleftharpoons[i^*]{\quad} \quad \mathcal{D} \quad \xrightleftharpoons{\quad} \quad \mathcal{X}$$

for which $\mathcal{D} = \mathcal{D}(R)$ for some ring R and $i^*(R)$ is an exceptional object of \mathcal{Y} .

Given such a recollement, the homological ring epimorphism is given by

$$R = \text{End}(R) \xrightarrow{\lambda} \text{End}(i^*(R)) =: S.$$

Up to equivalence, $\mathcal{Y} = \mathcal{D}(S)$, $i^* = - \otimes_R S$, $i_* = \lambda_*$ and $i^! = \mathbf{R}\text{Hom}_R(S, -)$.

Theorem. [27, Theorem 4.1] *Let R be a ring and Σ be a set of morphisms between finitely generated projective right R -modules. Then there exist a ring R_Σ and a morphism of rings $\lambda: R \rightarrow R_\Sigma$ such that*

- (1) λ is Σ -inverting, i.e. if $\alpha: P \rightarrow Q$ belongs to Σ , then $\alpha \otimes_R 1_{R_\Sigma}: P \otimes_R R_\Sigma \rightarrow Q \otimes_R R_\Sigma$ is an isomorphism of right R_Σ -modules, and
- (2) λ is universal Σ -inverting, i.e. if S is a ring such that there exists a Σ -inverting morphism $\psi: R \rightarrow S$, then there exists a unique morphism of rings $\bar{\psi}: R_\Sigma \rightarrow S$ such that $\bar{\psi} \lambda = \psi$.

The morphism $\lambda: R \rightarrow R_\Sigma$ is a ring epimorphism with $\text{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$. It is called the *universal localization of R at Σ* .

In general, a universal localization need not be a homological ring epimorphism, see [21] (and also [1, Example 5.4] for a different kind of example). For a hereditary ring R , however, $\lambda: R \rightarrow R_\Sigma$ is always a homological epimorphism, and R_Σ is a hereditary ring. In fact, it is even shown in [18, 6.1] that over hereditary rings universal localizations coincide with homological epimorphisms.

Let now \mathcal{U} be a set of finitely presented right R -modules of projective dimension one. For each $U \in \mathcal{U}$, consider a morphism α_U between finitely generated projective right R -modules such that

$$0 \rightarrow P \xrightarrow{\alpha_U} Q \rightarrow U \rightarrow 0$$

We will denote by $R_{\mathcal{U}}$ the universal localization of R at $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$. In fact, $R_{\mathcal{U}}$ does not depend on the class Σ chosen, cf. [7, Theorem 0.6.2], and we will also call it the *universal localization of R at \mathcal{U}* .

2.3. Classical tilting modules. Suppose R is a ring. Recall that an R -module T is said to be a *tilting module (of projective dimension at most one)* if the following conditions are satisfied:

- (1) $\text{proj.dim}(T) \leq 1$;
- (2) $\text{Ext}_R^1(T, T^{(I)}) = 0$ for each set I ; and
- (3) there is an exact sequence $0 \rightarrow R \rightarrow T_0 \rightarrow T_1 \rightarrow 0$ where T_0, T_1 belong to $\text{Add}T$.

The module T is called a *partial tilting module* if it satisfies the conditions (1) and (2). If, in addition, T is finitely presented, then we say that T is a *classical* (partial) tilting module.

It was shown in [1] that classical partial tilting modules induce recollements. In Theorem 2.5 below we state this result for the case of a hereditary ring, where there is an important connection to universal localizations.

2.4. Exceptional objects. Let us turn to the derived category $\mathcal{D} = \mathcal{D}(R)$. Recall that $X \in \mathcal{D}$ is *exceptional* if $\text{Hom}_{\mathcal{D}}(X, X[n]) = 0$ for all non-zero integers n . Further, the analog of finitely presented modules is provided by the *compact* objects, that is, the objects $X \in \mathcal{D}$ such that the functor $\text{Hom}_{\mathcal{D}}(X, -)$ preserves small coproducts, or equivalently, X is quasi-isomorphic to a bounded complex consisting of finitely generated projective modules. Of course, a finitely presented R -module over a hereditary ring R is exceptional if and only if it is a classical partial tilting module.

Let now $X \in \mathcal{D}$ be a compact exceptional object and denote by $\text{Tria } X$ the smallest full triangulated subcategory of \mathcal{D} which contains X and is closed under small coproducts. If $\text{Tria } X = \mathcal{D}$, then X is said to be a *tilting complex*. In general, we know from [16] that $\text{Tria } X$ is equivalent to the derived category $\mathcal{D}(C)$ of the endomorphism ring $C = \text{End}_{\mathcal{D}}(X)$ of X .

The following result characterizes the existence of a recollement of \mathcal{D} by derived categories of rings in terms of a suitable pair of exceptional objects.

Theorem 2.2. ([17], [22, 5.2.9], [23]) *There are rings R, B, C with a recollement of the form*

$$\mathcal{D}(B) \quad \begin{array}{c} \leftrightsquigarrow \\[-1ex] \rightrightarrows \end{array} \quad \mathcal{D}(R) \quad \begin{array}{c} \rightrightarrows \\[-1ex] \leftrightsquigarrow \end{array}^{j_!} \quad \mathcal{D}(C)$$

if and only if there are exceptional objects $X, Y \in \mathcal{D}(R)$ such that

- (i) X is compact,
- (ii) Y is a self-compact object, that is, $\text{Hom}_{\mathcal{D}}(Y, -)$ preserves small coproducts in $\text{Tria } Y$,
- (iii) $\text{Hom}_{\mathcal{D}}(X[n], Y) = 0$ for all $n \in \mathbb{Z}$,

- (iv) $X \oplus Y$ generates $\mathcal{D}(R)$, that is, an object $C \in \mathcal{D}(R)$ is zero whenever $\text{Hom}_{\mathcal{D}}(X \oplus Y, C[n]) = 0$ for every integer n .

In particular, $X = j_!(C)$, and $\text{Tria } X$ is equivalent to $\mathcal{D}(C)$.

2.5. Exceptional sequences. In this subsection, let A be a hereditary artin algebra with n non-isomorphic simple modules. Recall that a sequence of exceptional A -modules (X_1, X_2, \dots, X_m) is called an *exceptional sequence* if $\text{Hom}_A(X_j, X_i) = 0$ and $\text{Ext}_A^1(X_j, X_i) = 0$ for each pair $i < j$. An exceptional sequence (X_1, X_2, \dots, X_m) is called *complete* if $m = n$.

By Schur's lemma the endomorphism ring of a simple module is a skew-field. This is also the case for any indecomposable exceptional module of finite length by a result of Happel and Ringel.

Proposition 2.3. [15, 4.1 and 4.2] (1) If X_A is an indecomposable, finitely generated, and exceptional module, then the endomorphism ring of X is a skew-field.

(2) If X_A is a finitely generated, multiplicity-free, and exceptional A -module, then its indecomposable direct summands can be arranged into an exceptional sequence, which will be complete whenever X is a classical tilting A -module.

Theorem 2.4. [26, Theorem 4] Let A be a hereditary artin algebra, and T a multiplicity-free classical tilting A -module. Then the endomorphism rings of the indecomposable summands of T are precisely the endomorphism rings of the non-isomorphic simple modules.

The concept of (complete) exceptional sequence is available in the derived category $\mathcal{D}(A)$ as well. Since A is hereditary, the indecomposable objects in $\mathcal{D}(A)$ are the shifts of the indecomposable A -modules. Hence Proposition 2.3 holds in $\mathcal{D}(A)$, too. Given a compact, multiplicity-free, and exceptional object X in $\mathcal{D}(A)$, we can decompose X into a direct sum $Y_1[k_1] \oplus Y_2[k_2] \oplus \dots \oplus Y_s[k_s]$ such that the Y_i 's are A -modules and $k_1 < k_2 < \dots < k_s$. Since modules have no extensions in negative degrees, there are no nontrivial homomorphisms from $Y_i[k_i]$ to $Y_j[k_j]$ whenever $i > j$. Hence we can order the indecomposable direct summands of X into an exceptional sequence, which will be complete whenever X generates $\mathcal{D}(A)$ and therefore has n indecomposable direct summands.

Here the question arises, whether Theorem 2.4 holds for tilting complexes. This question will be answered positively later, in Corollary 5.2.

2.6. Perpendicular categories. Recollements are closely related to torsion theories and the outer terms in a recollement are equivalent to certain perpendicular categories, see [17, 22]. Perpendicular categories behave especially well in hereditary situations.

For any ring R and any R -module X , the *perpendicular category* \widehat{X} is by definition the full subcategory of $\text{Mod-}R$ consisting of the modules M satisfying $\text{Hom}_R(X, M) = 0$ and $\text{Ext}_R^1(X, M) = 0$.

The next result extends a result by Happel, Rickard and Schofield from finite dimensional hereditary algebras to semihereditary rings, and from module category level to derived category level. Recall that a ring is hereditary if all submodules of projective modules are again projective. If this is required only for finitely generated submodules, it is called *semihereditary*. For example, *von Neumann regular* rings, that is, rings R such that every element a can be written as $a = axa$ for some x in R (depending on a), are semihereditary. Commutative semihereditary domains are called *Prüfer domains*. The subring of \mathbb{C} consisting of the algebraic integers is a non-noetherian Prüfer domain of global dimension 2, cf. [10, VI, 4.5]. Another example of a ring that is semihereditary but not hereditary (on one side) can be found in [19, 2.33].

Theorem 2.5. Let R be a semihereditary ring, and X_R a finitely presented, exceptional R -module. Then there exists a ring B such that the following holds:

- (1) ([14, Proposition 3]) The perpendicular category \widehat{X} is equivalent to $\text{Mod-}B$.
(2) There is a recollement

$$\mathcal{D}(B) \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \qquad \longrightarrow \end{array} \quad \mathcal{D}(R) \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \qquad \longrightarrow \end{array} \quad \mathcal{D}(C)$$

where $C = \text{End}_R(X)$ is the endomorphism ring of X .

- (3) The ring B can be chosen as universal localization of R at X . Further, B is hereditary if so is R .

Proof. Since R is semihereditary, every finitely presented R -module has projective dimension ≤ 1 . The statements are thus contained in [1, Example 4.5 and Theorem 4.8], we give some details for the reader's convenience. By [1, Lemma 4.1] the perpendicular category \widehat{X} coincides with the essential image of the restriction functor λ_* given by the universal localization $\lambda : R \rightarrow B$ at X . Moreover, \widehat{X} is a reflective subcategory of $\text{Mod-}R$. This means by definition that every module $M \in \text{Mod-}R$ admits a \widehat{X} -reflection, that is, a morphism $\eta_M : M \rightarrow N$ such that $N \in \widehat{X}$ and $\text{Hom}_R(\eta_M, Y) : \text{Hom}_R(N, Y) \rightarrow \text{Hom}_R(M, Y)$ is bijective for all $Y \in \widehat{X}$.

First case: X is projective. $\lambda : R \rightarrow B$ can be chosen as universal localization at the zero map $\Sigma = \{\sigma : 0 \rightarrow X\}$. Hence it is a homological epimorphism, because R has weak global dimension bounded by 1, see [19, 4.67]. Then $\mathcal{D}(R)$ is a recollement of $\text{Tria } X \cong \mathcal{D}(C)$ and $\mathcal{D}(B)$, see [1, Example 4.5].

Note also that the R -module B_R is isomorphic to the \widehat{X} -reflection of R , and by [9, Section 1] the latter coincides with $R/\tau_X(R)$ where τ_X denotes the trace of X .

Second case: X has projective dimension one. The universal localization $\lambda : R \rightarrow B$ at X is a homological epimorphism by [19, 4.67]. Then $\mathcal{D}(R)$ is a recollement of $\text{Tria } X \cong \mathcal{D}(C)$ and $\mathcal{D}(B)$ by [20].

For later application, we give an explicit description of B as Bongartz complement of X , cf. [9, Section 1]: if c is the minimal number of generators of $\text{Ext}_R^1(X, R)$ as a module over $C = \text{End}_R(X)$, then there exists an exact sequence

$$E : 0 \rightarrow R \rightarrow M \rightarrow X^{(c)} \rightarrow 0$$

with the following properties:

- (1) any exact sequence $0 \rightarrow R \rightarrow N \rightarrow X \rightarrow 0$ has the form Ef for some $f \in \text{Hom}_R(X, X^{(c)})$,
- (2) $T = M \oplus X$ is a tilting module,
- (3) $M/\tau_X(M)$ is the \widehat{X} -reflection of R ,
- (4) $B \cong \text{End}_R(M)/\tau_X(M)$ as rings, and $B \cong M/\tau_X(M)$ as R -modules. \square

Proposition 2.6. *In the situation of Theorem 2.5, assume in addition that R is an artin algebra and X is indecomposable. Then the following hold true.*

- (1) *If X is projective, then B is an artin algebra, and the simple B -modules are precisely the simple R -modules that are not isomorphic to $X/\text{Rad}(X)$.*
- (2) *If X has projective dimension one, then B is an artin algebra, and viewed as an R -module, B complements X to a tilting module $T = B \oplus X$.*

Proof. (1) Let e be an idempotent such that $X = eR$. Then $B \cong R/ReR$ is an artin algebra. Moreover, $\widehat{X} = \{Y_R \mid Ye = 0\}$ is closed under submodules, so every simple B -module is also a simple R -module, and of course it is not isomorphic to $eR/e\text{Rad}(R)$. The converse implication is obvious.

(2) If X is indecomposable, then $C = \text{End}_R(X)$ is a skew-field by Proposition 2.3, and c is the C -dimension of $\text{Ext}_R^1(X, R)$, which is finite because X is finitely generated. Applying $\text{Hom}_R(X, -)$ to the universal sequence $0 \rightarrow R \rightarrow M \rightarrow X^c \rightarrow 0$, we obtain a long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(X, R) &\rightarrow \text{Hom}_R(X, M) \rightarrow \text{Hom}_R(X, X^c) \rightarrow \text{Ext}_R^1(X, R) \\ &\rightarrow \text{Ext}_R^1(X, M) \rightarrow \text{Ext}_R^1(X, X^c) \rightarrow 0 \end{aligned}$$

(recall that R is hereditary). The map $\text{Hom}_R(X, X^c) \rightarrow \text{Ext}_R^1(X, R)$ is bijective by construction, $\text{Hom}_R(X, R) = 0$ because X is not projective, and $\text{Ext}_R^1(X, X^c) = 0$ since X is exceptional. Therefore $M \in \widehat{X}$ is the \widehat{X} -reflection of R . Hence, as an R -module, $B \cong M$ is the Bongartz complement of X , and moreover, $B \cong \text{End}_R(M)$ is an artin algebra because M is finitely generated. \square

3. COMPLETING RECOLLEMENT DIAGRAMS

Proofs of classical Jordan Hölder theorems typically employ an argument called butterfly lemma, which helps to compare composition series of various (sub)objects. The results of this Section will serve a similar purpose for triangulated or derived categories.

Proposition 3.1. *Let A be a ring. Every diagram of the following form, involving a horizontal recollement and a vertical one,*

$$\begin{array}{ccccc} & & \mathcal{D}(E) & & \\ & & g \uparrow & h \uparrow & i \\ & & \downarrow & \downarrow & \\ \mathcal{D}(B) & \xleftarrow{\quad a \quad} & \mathcal{D}(A) & \xleftarrow{\quad d \quad} & \mathcal{D}(C) \\ \xleftarrow{\quad b \quad} & & \xleftarrow{\quad e \quad} & & \\ \xleftarrow{\quad c \quad} & & \xleftarrow{\quad f \quad} & & \\ & & k \uparrow & l \uparrow & m \\ & & \downarrow & \downarrow & \\ & & \mathcal{D}(F) & & \end{array}$$

can be completed to a diagram of the following form, involving two horizontal and two vertical recollements,

$$\begin{array}{ccccc} \mathcal{D}(B) & \xleftarrow{\quad u \quad} & \mathcal{D}(G) & \xleftarrow{\quad r \quad} & \mathcal{D}(E) \\ \xleftarrow{\quad v \quad} & \xleftarrow{\quad w \quad} & \xleftarrow{\quad s \quad} & \xleftarrow{\quad t \quad} & \\ 1 \uparrow & 1 \uparrow & x \uparrow & z \uparrow & g \uparrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathcal{D}(B) & \xleftarrow{\quad a \quad} & \mathcal{D}(A) & \xleftarrow{\quad d \quad} & \mathcal{D}(C) \\ \xleftarrow{\quad b \quad} & & \xleftarrow{\quad e \quad} & & \\ \xleftarrow{\quad c \quad} & & \xleftarrow{\quad f \quad} & & \\ & & kf \uparrow & el \uparrow & k \uparrow \\ & & \downarrow & \downarrow & \downarrow \\ & & md & & m \\ & & \mathcal{D}(F) & \xleftarrow{\quad 1 \quad} & \mathcal{D}(F) \\ & & \xleftarrow{\quad -1 \quad} & & \\ & & \downarrow & & \end{array}$$

where G is a differential graded algebra.

Proof. First we fill in the bottom square in the diagram by putting all functors from $\mathcal{D}(F)$ to itself to be identity, and by using the compositions $m \cdot d$, $e \cdot l$ and $k \cdot f$ to connect $\mathcal{D}(A)$ and $\mathcal{D}(F)$. Next we can complete by [25, Theorem 2.4(b)] the half-recollement involving $\mathcal{D}(A)$ and $\mathcal{D}(F)$ to get some triangulated category in the middle of the top row. By [22, 4.3.6, 4.4.8], $z(A)$ is a compact generator of this triangulated category, and hence by [17, Theorem 4.3] this is equivalent to the derived category of the differential graded endomorphism algebra G of $z(A)$.

Now we complete the upper left square: The embedding functor v exists, since the composition $1_B \cdot (b \cdot e) \cdot l$ vanishes and thus the image of b must be inside $\mathcal{D}(G)$. The right adjoint w is defined by $w := y \cdot c \cdot 1_B$, and the composition satisfies $v \cdot w = v \cdot y \cdot c \cdot 1_B = 1_B \cdot b \cdot c \cdot 1_B = 1_B$. Similarly, the left adjoint u is defined by $u := y \cdot a \cdot 1_B$ and composition satisfies $v \cdot u = v \cdot y \cdot a \cdot 1_B = 1_B \cdot b \cdot a \cdot 1_B = 1_B$.

To complete the upper right square, we define s be $s := y \cdot e \cdot g (= y \cdot e \cdot i)$. Then $r := h \cdot d \cdot z$ is a left adjoint: The image of $h \cdot d$ is contained in the kernel of $e \cdot l$ and thus in the image of the embedding y . Therefore, $r \cdot s = h \cdot d \cdot z \cdot y \cdot e \cdot g = h \cdot d \cdot e \cdot g = h \cdot g = 1_E$. Similarly, $t := h \cdot f \cdot x$ is a right adjoint: Again, the image of $h \cdot f$ is contained in the image of the embedding y , and therefore $t \cdot s = h \cdot f \cdot x \cdot y \cdot e \cdot g = h \cdot f \cdot e \cdot g = h \cdot g = 1_E$.

Finally, we check that the top row really forms a recollement. By definition the functors v , r and t are full embeddings. The composition $v \cdot s$ vanishes, since $v \cdot s = v \cdot y \cdot e \cdot g = 1_B \cdot b \cdot e \cdot g = 0$. The kernel of s is indeed the kernel of $y \cdot e$, so it is precisely $\mathcal{D}(B)$. It remains to check the existence of the canonical triangles. Let X be in an object in $\mathcal{D}(G)$ and write it as middle term of a canonical triangle (for the given recollement in the second row) $Y \rightarrow X \rightarrow Z \rightsquigarrow$, where Y is in $\mathcal{D}(B)$ and Z is in $\mathcal{D}(C)$. Since X is in $\mathcal{D}(G)$, its e -image Z is in the kernel of l and thus it is in $\mathcal{D}(E)$, as required. Thus the given triangle also is a canonical triangle for the first row. The second canonical triangle for the second row, for the given object X , is $U \rightarrow X \rightarrow V \rightsquigarrow$ with U in $\mathcal{D}(C)$ and V in $\mathcal{D}(B)$. This triangle serves as a canonical triangle for the first row as well, once we have shown that U is already in $\mathcal{D}(E)$: The object U is obtained from X by applying $y \cdot e$, and thus the image of U under l is the image of X under $y \cdot e \cdot l = s \cdot h \cdot l = 0$. \square

Dually one can prove the following.

Proposition 3.2. *Let A be an algebra. Every diagram of the following form, involving a horizontal recollement and a vertical one,*

$$\begin{array}{ccccc}
& & \mathcal{D}(E) & & \\
& \uparrow & | & \uparrow & \\
g & \left| \begin{matrix} h \\ \downarrow \\ i \end{matrix} \right. & & & \\
& \downarrow & & & \\
\mathcal{D}(B) & \xleftarrow{\quad a \quad} & \mathcal{D}(A) & \xleftarrow{\quad d \quad} & \mathcal{D}(C) \\
& \xleftarrow{\quad b \quad} & & \xleftarrow{\quad e \quad} & \\
& \xleftarrow{\quad c \quad} & & \xleftarrow{\quad f \quad} & \\
& \uparrow & | & \uparrow & \\
k & \left| \begin{matrix} l \\ \downarrow \\ m \end{matrix} \right. & & & \\
& \downarrow & & & \\
& & \mathcal{D}(F) & &
\end{array}$$

can be completed to a diagram of the following form, involving two horizontal and two vertical recollements,

$$\begin{array}{ccccc}
 & \xleftarrow{\quad 1 \quad} & & \xleftarrow{\quad 1 \quad} & \\
 \mathcal{D}(E) & \xrightarrow{\quad -1 \rightarrow \quad} & & \mathcal{D}(E) & \\
 & \xleftarrow{\quad 1 \quad} & & \xleftarrow{\quad cg \quad} & \\
 g \uparrow \begin{matrix} | & | \\ h & i \end{matrix} \downarrow & & cg \uparrow \begin{matrix} | & | \\ hb & ai \end{matrix} \downarrow & & \\
 \mathcal{D}(B) & \xrightarrow{\quad a \quad} & \mathcal{D}(A) & \xrightarrow{\quad d \quad} & \mathcal{D}(C) \\
 & \xleftarrow{\quad b \rightarrow \quad} & & \xleftarrow{\quad e \rightarrow \quad} & \\
 & \xleftarrow{\quad c \quad} & & \xleftarrow{\quad f \quad} & \\
 k \uparrow \begin{matrix} | & | \\ l & m \end{matrix} \downarrow & & \uparrow \begin{matrix} | & | \\ \cdot & \cdot \end{matrix} \downarrow & & 1 \uparrow \begin{matrix} | & | \\ 1 & 1 \end{matrix} \downarrow \\
 \mathcal{D}(F) & \xrightleftharpoons{\quad \quad} & \mathcal{X} & \xrightleftharpoons{\quad \quad} & \mathcal{D}(C)
 \end{array}$$

where \mathcal{X} is a triangulated category.

It follows from the proposition that the triangulated category \mathcal{X} has a recollement structure filtered by the two derived module categories $\mathcal{D}(F)$ and $\mathcal{D}(C)$. But in general we don't know whether \mathcal{X} itself is equivalent to a derived module category.

For hereditary artin algebras, we are now able to generalize Theorem 2.5 to an exceptional and compact complex X .

Corollary 3.3. *Let A be a hereditary artin algebra, and X an exceptional and compact complex in $\mathcal{D}(A)$. Then there exists a hereditary artin algebra B with a homological ring epimorphism $A \rightarrow B$ and a recollement*

$$\mathcal{D}(B) \quad \xrightleftharpoons{\quad \quad} \quad \mathcal{D}(A) \quad \xrightleftharpoons{\quad \quad} \quad \mathcal{D}(C)$$

where $C = \text{End}_A(X)$ is the endomorphism ring of X .

Proof. Assume X is multiplicity free. By Subsection 2.5, the indecomposable direct summands of X can be ordered into an exceptional sequence, say (X_1, X_2, \dots, X_s) , in $\mathcal{D}(A)$. For each pair $i < j$, there is no nontrivial homomorphism from X_j to X_i . Each X_i is a shift of an indecomposable, finitely presented, exceptional A -module, so we apply Theorem 2.5 and Proposition 2.6 iteratively to X_s, X_{s-1}, \dots , and X_1 . In the first step we obtain a hereditary artin algebra B_s such that we have a recollement

$$\mathcal{D}(B_s) \quad \xrightleftharpoons{\quad \quad} \quad \mathcal{D}(A) \quad \xrightleftharpoons{\quad \quad} \quad \mathcal{D}(C_s)$$

where $C_s = \text{End}_A(X_s)$ is the endomorphism ring of X_s , and B_s is a universal localization of A . Now X_1, \dots, X_{s-1} belong to $\mathcal{D}(B_s)$. Applying Theorem 2.5 to X_{s-1} and B_s , we obtain a hereditary artin algebra B_{s-1} such that

$$\mathcal{D}(B_{s-1}) \quad \xrightleftharpoons{\quad \quad} \quad \mathcal{D}(B_s) \quad \xrightleftharpoons{\quad \quad} \quad \mathcal{D}(C_{s-1})$$

where $C_{s-1} = \text{End}_{B_s}(X_{s-1}) = \text{End}_A(X_{s-1})$ is the endomorphism ring of X_{s-1} , and B_{s-1} is a universal localization of B_s . Now we are in the situation of Proposition 3.2. By completing the recollements we obtain a new recollement of $\mathcal{D}(A)$ filtered by $\mathcal{D}(B_{s-1})$ and some triangulated category \mathcal{X} , which again admits a recollement filtered by $\mathcal{D}(C_{s-1})$ and $\mathcal{D}(C_s)$. By construction, $\mathcal{D}(C_i) \cong \text{Tria } X_i$ for $i = s, s-1$, hence $\mathcal{X} \cong \text{Tria } (X_s \oplus X_{s-1}) \cong \mathcal{D}(\tilde{C})$ for $\tilde{C} = \text{End}_A(X_s \oplus X_{s-1})$. Moreover, the composition $A \rightarrow B_s \rightarrow B_{s-1}$ is a homological epimorphism. To finish the proof we just have to continue iteratively. \square

In the situation of Proposition 3.1, the image of F under the functor md is always exceptional and compact. Thanks to the corollary, if A is a hereditary artin algebra, we can choose G to be also hereditary and artin. This fact will be used later in the inductive proof of our main result Theorem 5.1.

4. LIFTING AND RESTRICTING RECOLLEMENTS, DERIVED SIMPLICITY

In the literature, various kinds of recollements are used for different purposes; these involve bounded, left or right bounded or unbounded derived categories, homotopy categories of projectives. Although we are focussing on unbounded derived categories in the main part of this article, we collect in this Section some information on comparing recollements of different types. Roughly speaking, lifting to 'larger' categories always is possible, while restricting to 'smaller' categories is problematic. We do not provide a final answer to the problem, whether the existence of a recollement always implies the existence of another one that can be restricted. Nor do we solve the question, which functors in an existing recollement do restrict.

Let A , B and C be any rings. Recall that $\text{Mod-}A$ denotes the category of arbitrary right A -modules, and write $\text{Proj-}A$ for the full subcategory of all projective modules. We ask for relations between the following recollements:

$$\begin{array}{ccccccc} (R0) & K^b(\text{Proj-}B) & \xrightleftharpoons{\quad} & K^b(\text{Proj-}A) & \xrightleftharpoons{\quad} & K^b(\text{Proj-}C) \\ (R1) & \mathcal{D}^b(B) & \xrightleftharpoons{\quad} & \mathcal{D}^b(A) & \xrightleftharpoons{\quad} & \mathcal{D}^b(C) \\ (R2) & \mathcal{D}^-(B) & \xrightleftharpoons{\quad} & \mathcal{D}^-(A) & \xrightleftharpoons{\quad} & \mathcal{D}^-(C) \\ (R3) & \mathcal{D}(B) & \xrightleftharpoons{\quad} & \mathcal{D}(A) & \xrightleftharpoons{\quad} & \mathcal{D}(C) \end{array}$$

Lemma 4.1. *If the ring A has a recollement of the form (R1), then it has a recollement of the form (R2). The converse holds true if A has finite global dimension.*

Proof. See [17, Corollary 6], [24, Theorem 2]: From the recollement (R1), the complexes $Y = i_*(B)$, $X = j_!(C) \in K^b(\text{Proj-}A)$ provide the required candidates for the existence of a recollement of the form (R2). Note that $\text{Hom}_{\mathcal{D}(A)}(Y, Y[n]^{(I)}) \cong \text{Hom}_{\mathcal{D}(B)}(B, B[n]^{(I)}) = 0$ for all non-zero integers n and all sets I , because i_* is a full embedding. For the converse, see [17, Theorem 7]. \square

The same argument as above also proves the following lifting of recollements. The second statement follows from [17, Proposition 4].

Lemma 4.2. *If the ring A has a recollement of the form (R0), then it has a recollement of the form (R2). The converse holds true if A has finite global dimension.*

Lemma 4.3. *If the ring A has a recollement of the form (R2), then it has a recollement of the form (R3).*

Proof. Given a recollement of the form (R2), we know from [17, Theorem 1], [24, Theorem 2] that $X = j_!(C)$ is compact exceptional, $Y = i_*(B)$ is self-compact and exceptional, and $X \oplus Y$ generates $\mathcal{D}(A)$. So, by Theorem 2.2 there exists a recollement of the form (R3). \square

Lemma 4.4. *If the ring A has a recollement of the form (R3) and $Y = i_*(B)$ belongs to $K^b(\text{Proj-}A)$, then A has a recollement of the form (R2).*

In particular, if a hereditary artin algebra A has a recollement of the form (R3), then it has a recollement of the form (R2).

Proof. Given a recollement of the form (R3), we know from Theorem 2.2 that $X = j_!(C)$ is a compact exceptional object, and $Y = i_*(B)$ is self-compact. Then the statement follows from the criterion in [17, Theorem 1], [24, Theorem 2], since we have as in the proof of Lemma 4.1 that $\text{Hom}_{\mathcal{D}(A)}(Y, Y[n]^{(I)}) = 0$ for all non-zero integers n and all sets I .

If A is a hereditary artin algebra, we can assume by Corollary 3.3 that the recollement of the form (R3) is induced by a homological ring epimorphism $\lambda : A \rightarrow B$ to a hereditary artin algebra B , and i_* is the canonical embedding $\mathcal{D}(B) \rightarrow \mathcal{D}(A)$. So $Y = i_*(B) = B$ is an A -module and thus belongs to $K^b(\text{Proj-}A)$. \square

Corollary 4.5. *For a hereditary artin algebra A , the following assertions are equivalent:*

- (1) *A has a recollement of the form (R0);*
- (2) *A has a recollement of the form (R1);*
- (3) *A has a recollement of the form (R2);*
- (4) *A has a recollement of the form (R3).*

Proof. Combine Lemmas 4.1 – 4.4. \square

Example 4.6. *The following example from [17, Example 8] provides a recollement of the form (R2) which does not restrict to \mathcal{D}^b -level. Indeed, by [17, Proposition 4], a recollement of the form (R2) restricts to a recollement of the form (R1) if and only if the functor $j_! : \mathcal{D}^-(C) \rightarrow \mathcal{D}^-(A)$ restricts to \mathcal{D}^b -level - a condition that fails here.*

Let A be the finite dimensional algebra over a field k given by

$$1 \cdot \begin{array}{c} \xleftarrow{\alpha} \\[-1ex] \xrightarrow{\beta} \end{array} \cdot 2 \quad [\beta\alpha\beta = 0].$$

The simple module $S(1)$ and the projective module $P(2)$ provide a recollement of A of the form (R2) with $B = \text{End}_A(S(1)) \cong k$ and $C = \text{End}_A(P(2)) \cong k[x]/(x^2)$. By construction, the functor $j_!$ is given by the left derived functor $- \otimes_C^L P(2)$. It cannot restrict to \mathcal{D}^b -level, for $P(2)$ as a left C -module has infinite projective dimension.

For the rest of the section, we focus on rings that do not admit non-trivial recollements as above. The following definition slightly extends a definition of Wiedemann [28], who considered bounded derived categories only.

Definition 4.7. A ring R is called *derived simple* if $\mathcal{D}(R)$ does not admit any non-trivial recollement whose factors are derived categories of rings.

A ring R is called *derived simple* with respect to \mathcal{D}^* if $\mathcal{D}^*(R)$ for $* = \{b, +, -\}$ does not admit any non-trivial recollement whose factors are derived categories (of the form \mathcal{D}^*) of rings.

In Section 6 we will see why it is necessary to require the factors to be derived categories of rings again, and not just triangulated categories. Observe that \mathcal{D}^- -derived simplicity implies \mathcal{D}^b -derived simplicity by Lemma 4.1, and \mathcal{D} -derived simplicity implies \mathcal{D}^- -derived simplicity by Lemma 4.3. As we will see in [2], the converse does not hold for general rings. However the situation is fine for hereditary artin algebras.

Corollary 4.8. *For a hereditary artin algebra A , the following assertions are equivalent:*

- (1) *A is derived simple.*

- (2) A is \mathcal{D}^- -derived simple.
- (3) A is \mathcal{D}^b -derived simple.

Proof. It follows from Corollary 4.5. \square

Derived simple rings obviously satisfy a Jordan Hölder theorem for derived categories. Computations or proofs in K-theory or about homological dimensions that are based on recollements, that is on an induction on the length of a stratification, need to be based on the case of derived simple rings. Hence it is of interest to identify such rings or classes thereof.

Lemma 4.9. *A semiperfect ring R is derived simple provided that for each finitely generated projective R -module P the trace of P in R equals R .*

Proof. The condition means that $\text{add}R = \text{add}P$ for any finitely generated projective R -module P . So, for any two non-zero finitely generated projective R -modules P, Q there is a non-zero homomorphism $P \rightarrow Q$ which maps an indecomposable summand of P isomorphically onto an indecomposable summand of Q , and is zero elsewhere. Hence a compact complex (of finitely generated projective R -modules) must have self-extensions unless it is just a projective module, up to shift. Then all compact exceptional complexes generate $\mathcal{D}(R)$ and are tilting complexes. By Theorem 2.2 it follows that R is derived simple. \square

Proposition 4.10. *Let R be a semihereditary ring. Then the following assertions are equivalent.*

- (1) R is derived simple.
- (2) Every non-zero finitely presented exceptional module is tilting.
- (3) The universal localization $\lambda : R \rightarrow R_T$ at any non-zero finitely presented exceptional R -module T vanishes.

If R satisfies these conditions, then for each finitely generated projective R -module P the trace of P in R equals R .

Proof. (1) \Rightarrow (2): By Theorem 2.5, every non-zero finitely presented exceptional module X gives rise to a recollement

$$\mathcal{D}(B) \quad \begin{array}{c} \longleftarrow \\[-1ex] \longrightarrow \end{array} \quad \mathcal{D}(A) \quad \begin{array}{c} \longleftarrow \\[-1ex] \longrightarrow \end{array} \quad \mathcal{D}(C)$$

where $C = \text{End}_R(X)$. So $\mathcal{D}(C) \cong \text{Tria } X \neq 0$, hence $\mathcal{D}(B) = 0$. But this means that X generates $\mathcal{D}(R)$ and is thus a tilting module.

(2) \Rightarrow (3) follows immediately from Theorem 2.5.

(3) \Rightarrow (1): Given a recollement

$$\mathcal{D}(B) \quad \begin{array}{c} \longleftarrow \\[-1ex] \longrightarrow \end{array} \quad \mathcal{D}(A) \quad \begin{array}{c} \xleftarrow{j_!} \\[-1ex] \xrightarrow{j^*} \end{array} \quad \mathcal{D}(C)$$

with $\mathcal{D}(C) \neq 0$, we know from Theorem 2.2 that $X = j_!(C)$ is a non-zero compact exceptional object, hence a direct sum of shifts of finitely presented, exceptional modules. Let T be one of these modules. By Theorem 2.5, there is a recollement of $\mathcal{D}(R)$ by $\text{Tria } T$ and $\mathcal{D}(R_T)$ which must be trivial by condition (3). Since $\text{Tria } X$ contains $\text{Tria } T$, it follows that the recollement above must also be trivial.

Finally, the additional statement is condition (3) in the special case when T is projective, cf. the first case in Theorem 2.5. \square

Now we obtain several examples.

Proposition 4.11. (1) All local rings are derived simple.

- (2) A right artinian hereditary ring is derived simple if and only if it is simple artinian.
- (3) A commutative semihereditary ring is derived simple if and only if it is a Prüfer domain. In particular, \mathbb{Z} and polynomial rings in one variable over fields are derived simple.
- (4) A von Neumann regular ring R is derived simple if and only if for each finitely generated projective module P , the trace of P in R equals R .

Proof. (1) follows immediately from Lemma 4.9.

(2) Simple artinian rings satisfy the criterion in Lemma 4.9 and are therefore derived simple. Conversely, if a right artinian hereditary ring is derived simple, then we know from Proposition 4.10 that all indecomposable finitely generated projective modules are tilting, which shows that there is just one projective module up to isomorphism, which must of course be simple. Then R is simple artinian.

(3) As shown in [19, 2.44], if P is a finitely generated projective module over a commutative ring R , then $R = \tau_P(R) \oplus \text{ann}_R(P)$, where $\text{ann}_R(P) = \{r \in R \mid xr = 0 \text{ for all } x \in P\}$. So, every commutative semihereditary ring R which is derived simple must be a domain. In fact, if $x \in R \setminus \{0\}$, then by assumption $P = xR$ is a finitely generated projective module, so we infer from Proposition 4.10 that $\text{ann}_R(x) = \text{ann}_R(P) = 0$. Conversely, every Prüfer domain R satisfies condition (3) in Proposition 4.10. Indeed, if T is a non-zero finitely presented exceptional R -module, then T is projective by [8, 2.2]. Furthermore, since T is faithful, $R = \tau_T(R)$, so the universal localization at T is trivial.

(4) Recall that R is a semihereditary ring of weak global dimension zero [19, 2.32 and 4.21]. So, the only-if-part follows from Proposition 4.10. Moreover, all finitely presented modules are finitely generated projective. But then every non-zero compact exceptional object in $\mathcal{D}(R)$ is a direct sum of shifts of finitely generated, projective modules. Thus we can prove the if-part arguing as in Proposition 4.10, (3) \Rightarrow (1). \square

Wiedemann [28] has shown that derived simplicity with respect to \mathcal{D}^b is a non-trivial property for finite dimensional algebras, going much beyond local algebras; he found an algebra with two simple modules that is derived simple. Happel [12] even showed derived simplicity, also with respect to \mathcal{D}^b , for a series of algebras with two simple modules and of finite global dimension.

5. A JORDAN HÖLDER THEOREM FOR DERIVED CATEGORIES OF HEREDITARY ARTIN ALGEBRAS

In this section we state and prove the main result of this article, which covers hereditary artin algebras and algebras which are derived equivalent to them.

Here by a *stratification* of the derived category $\mathcal{D}(A)$ of a ring A we mean a sequence of iterated recollements of the following form: a recollement of A , if it is not derived simple,

$$\mathcal{D}(B) \quad \begin{array}{c} \xleftarrow{\quad} \\[-1ex] \xrightarrow{\quad} \end{array} \quad \mathcal{D}(A) \quad \begin{array}{c} \xleftarrow{\quad} \\[-1ex] \xrightarrow{\quad} \end{array} \quad \mathcal{D}(C)$$

and a recollement of B , if it is not derived simple,

$$\mathcal{D}(B_1) \quad \begin{array}{c} \xleftarrow{\quad} \\[-1ex] \xrightarrow{\quad} \end{array} \quad \mathcal{D}(B) \quad \begin{array}{c} \xleftarrow{\quad} \\[-1ex] \xrightarrow{\quad} \end{array} \quad \mathcal{D}(B_2)$$

and a recollement of C , if it is not derived simple,

$$\mathcal{D}(C_1) \quad \begin{array}{c} \xleftarrow{\quad} \\[-1ex] \xrightarrow{\quad} \end{array} \quad \mathcal{D}(C) \quad \begin{array}{c} \xleftarrow{\quad} \\[-1ex] \xrightarrow{\quad} \end{array} \quad \mathcal{D}(C_2)$$

and recollements of B_i and of C_i ($i = 1, 2$), if they are not derived simple, and so on, until we arrive at derived simple rings at all positions, or continue ad infinitum.

Theorem 5.1. Let A be derived equivalent to a hereditary artin algebra and let S_1, \dots, S_n be representatives of the isomorphism classes of simple A -modules. Denote the endomorphism rings by $D_i := \text{End}_A(S_i)$. Then $\mathcal{D}(A)$ has a stratification whose n factors are the categories $\mathcal{D}(D_i)$. Any stratification of $\mathcal{D}(A)$ has precisely these factors, up to ordering and derived equivalence.

Note that derived equivalence for the skew-fields D_i just means Morita equivalence.

Proof. Without loss of generality, we assume A is hereditary. We will proceed by induction on the number n of isomorphism classes of non-isomorphic simple modules of the algebra A .

Any hereditary artin algebra has a standard stratification of length n whose factors are precisely the endomorphism ring of the simple modules. Indeed, a simple projective module S_1 generates an ideal J_1 that is projective on both sides - more precisely it is a heredity ideal and thus a stratifying ideal - and hence there is a recollement involving A , $D_1 = \text{End}_A(S_1)$ and A/J_1 , which again is a hereditary artin algebra, with simples S_2, \dots, S_n .

For uniqueness we will prove a stronger result by induction using Proposition 3.1: If A is a hereditary artin algebra, any stratification of $\mathcal{D}(A)$ can be rearranged into a finite chain of increasing derived module categories of hereditary artin algebras

$$\mathcal{D}(A_n) \iff \mathcal{D}(A_{n-1}) \iff \dots \iff \mathcal{D}(A_2) \iff \mathcal{D}(A_1)$$

of length n , where $A_1 = A$. Moreover this chain is induced by a sequence of homological epimorphisms $A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_{n-1} \rightarrow A_n$, such that A_n is derived simple and for each $1 \leq i \leq n-1$,

$$\mathcal{D}(A_{i+1}) \iff \mathcal{D}(A_i)$$

can be completed to a full recollement with the third term being the derived module category of some derived simple algebra, say, E_i . We write $E_n = A_n$ for convenience. These E_i 's are the endomorphism rings of simple A -modules and $\mathcal{D}(E_i)$ ($i = 1, \dots, n$) are precisely the derived simple factors in the original stratification.

When $n = 1$, the hereditary algebra A has only one simple module. This simple module is also projective, so A is Morita equivalent to a skew-field, and hence derived simple, cf. Proposition 4.11. In the following we assume $n \geq 2$. Suppose a stratification of $\mathcal{D}(A)$ starts with

$$(1) \quad \mathcal{D}(B) \iff \mathcal{D}(A) \iff \mathcal{D}(C)$$

and, if C is not derived simple,

$$(2) \quad \mathcal{D}(C_1) \iff \mathcal{D}(C) \iff \mathcal{D}(C_2).$$

Applying Proposition 3.1, we can rearrange the two recollements into

$$(3) \quad \mathcal{D}(B') \iff \mathcal{D}(A) \iff \mathcal{D}(C_2)$$

and

$$(4) \quad \mathcal{D}(B) \iff \mathcal{D}(B') \iff \mathcal{D}(C_1)$$

where $C_1 = E$, $C_2 = F$ and $B' = G$ in the notation of Proposition 3.1. Note that under the rearrangement the factors $\mathcal{D}(B)$, $\mathcal{D}(C_1)$ and $\mathcal{D}(C_2)$ are preserved. By Theorem 2.2, the image $j_!(C_2)$ of C_2 under the full embedding on the upper right corner of the recollement (3) is compact and exceptional in $\mathcal{D}(A)$. By Corollary 3.3 we can then assume that B' is a hereditary artin algebra and the recollement (3) is induced by a homological epimorphism $A \rightarrow B'$. Using the same argument on the recollement (4) we can assume that B is a hereditary artin algebra and the recollement (4) is induced by a homological epimorphism $B' \rightarrow B$.

The original stratification of $\mathcal{D}(A)$ is given by the recollement (1) and a stratification on $\mathcal{D}(B)$ and on $\mathcal{D}(C)$ respectively. By iterating the above procedure we can transport the recollements in the stratification on $\mathcal{D}(C)$ to the left hand side of $\mathcal{D}(A)$ and thus obtain a chain of increasing derived module categories of hereditary artin algebras

$$\mathcal{D}(B) \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} \quad \mathcal{D}(B'_1) \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} \quad \mathcal{D}(B'_2) \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} \quad \cdots \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} \quad \mathcal{D}(A)$$

induced by a sequence of homological epimorphisms $A \rightarrow \dots \rightarrow B'_2 \rightarrow B'_1 \rightarrow B'_0 = B$. The subfactors in the chain are precisely the derived simple factors in the stratification of $\mathcal{D}(C)$. Moreover, each B'_i is a partial tilting module over B'_{i+1} ($i \geq 0$). Hence the number of non-isomorphic simple modules of B'_i is strictly smaller than that of B'_{i+1} . This implies that the above chain, and hence the stratification of $\mathcal{D}(C)$, must have finite length.

Since B is a hereditary artin algebra and has a smaller number of non-isomorphic simple modules than A , we can apply induction and assume the stratification on $\mathcal{D}(B)$ has been rearranged as desired. Combining this with the chain obtained in the previous paragraph, we can rearrange the original stratification on $\mathcal{D}(A)$ into a finite chain of increasing derived module categories of hereditary artin algebras

$$\mathcal{D}(A_m) \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} \quad \cdots \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} \quad \mathcal{D}(A_2) \quad \begin{array}{c} \longleftrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} \quad \mathcal{D}(A_1)$$

of length, say, m , induced by a sequence of homological epimorphisms $A = A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_m$, and such that the factors $\mathcal{D}(E_i)$ ($i = 1, \dots, m$) are precisely the derived simple factors in the original stratification on $\mathcal{D}(A)$.

Consider the first recollement

$$\mathcal{D}(A_2) \quad \begin{array}{c} \Longleftrightarrow \\ \Longleftarrow \\ \Longleftarrow \end{array} \quad \mathcal{D}(A_1) \quad \begin{array}{c} \xrightarrow{j_!} \\ \Longleftrightarrow \\ \Longleftarrow \end{array} \quad \mathcal{D}(E_1)$$

taken from the right hand side of the above filtration. By Theorem 2.2, $X = j_!(E_1)$ is a compact exceptional object in $\mathcal{D}(A)$. We claim that X is indecomposable. Indeed, as explained in Subsection 2.5, the indecomposable summands of X can be arranged into an exceptional sequence. Therefore, X has a triangular (directed) endomorphism ring $\text{End}_A(X) \simeq E_1$. So E_1 has a simple projective module, generating a stratifying ideal J and thus inducing a recollement for E_1 . But E_1 is derived simple and the recollement must be trivial. Thus $J = E_1$ and E_1 is a simple algebra, which implies the claim.

Now since X is an indecomposable, finitely presented and exceptional module, by Theorem 2.5 A_2 can be chosen to be the hereditary artin algebra obtained from universal localization of A_1 at X . Note that A_2 has $n - 1$ simple modules by Proposition 2.6. By induction hypothesis, we see that $m = n$ and that the derived simple algebras E_2, \dots, E_n in the stratification are the endomorphism rings of the simple A_2 -modules.

If X is projective, $X/\text{Rad}(X)$ is a simple A -module. Therefore up to renumbering we have $\text{End}_A(X/\text{Rad}(X)) \simeq D_1$. Since $\text{End}_A(X/\text{Rad}(X)) \simeq \text{End}_A(X)$, we have $E_1 \simeq D_1$. The simple A_2 -modules are precisely those simple A -modules that are not isomorphic to $X/\text{Rad}(X)$, so $\{E_1 \simeq \text{End}_A(X), E_2, \dots, E_n\} = \{D_1, \dots, D_n\}$. If X has projective dimension one, then $(A_2)_A$ complements X to a tilting module $T = A_2 \oplus X$, so by Theorem 2.4 the endomorphism rings of the indecomposable summands of T are precisely D_1, \dots, D_n . As the endomorphism rings of the indecomposable summands of $(A_2)_A$ coincide with the endomorphism rings of the simple A_2 -modules, we conclude also in this case that $\{E_1 \simeq \text{End}_A(X), E_2, \dots, E_n\} = \{D_1, \dots, D_n\}$. \square

We are now ready to answer the question that has been stated after Theorem 2.4, about the endomorphism rings of indecomposable direct summands in a tilting complex. Recall that an

object T in $\mathcal{D}(A)$ is called a *tilting complex* if T is compact, exceptional, and $\mathcal{D}(A)$ equals $\text{Tria } T$, the smallest triangulated category containing T and closed under small coproducts.

Corollary 5.2. *Let A be a hereditary artin algebra, and T a multiplicity free tilting complex in $\mathcal{D}(A)$. Then the endomorphism rings of the indecomposable direct summands of T are precisely those of the non-isomorphic simple modules.*

Proof. By (2.5), the indecomposable direct summands of T form a complete exceptional sequence, say (T_1, T_2, \dots, T_n) . From the proof of Corollary 3.3, this exceptional sequence induces a stratification of $\mathcal{D}(A)$ whose factors are the derived module categories of $\text{End}_A(T_i)$ ($i = 1, 2, \dots, n$). Due to Theorem 5.1, these endomorphism rings are up to derived equivalence the endomorphism rings of the non-isomorphic simple A -modules. But for skew-fields, derived equivalence implies Morita equivalence. The T_i being indecomposable then implies their endomorphism rings are local and hence isomorphic to those of the simple modules S_i . \square

6. WHAT CAN FAIL

In this Section we first explain why only derived categories of rings should be permitted as outer terms of recollements in our context. We also give an example showing that Theorem 5.1 fails without finiteness assumptions - while we do not have examples of failure for artin algebras in general, that is, when dropping the assumption 'hereditary'.

Which kind of recollements is meaningful when trying to prove a Jordan Hölder theorem for derived categories of rings? Since recollement is a natural concept for triangulated categories in general, a natural first choice is to admit all triangulated categories as terms in a recollement. This choice, however, leads to an abundance of recollements, for instance in the following way: By the second Theorem in [1, 1.6], there exists a recollement of the derived category $\mathcal{D}(R)$ as soon as there exists an object T_1 generating a smashing subcategory. Thus, we may for instance choose T_1 to be a finitely generated R -module. Then we will get a recollement, where on the right hand side we get the triangulated category generated by T_1 . Under some assumptions (see [1]) this category is equivalent to the derived category of the differential graded endomorphism algebra of T_1 - which is an ordinary algebra only if T_1 has no self-extensions. We always get the derived category of another differential graded algebra on the left hand side. Hence, making such a generous choice for factors of recollements will imply that there are few derived simple rings, and it will move the question of derived simplicity to different kinds of triangulated categories. When considering recollements on this general level, the terms in a 'composition series' of $\mathcal{D}(R)$ usually will be triangulated categories that are much less accessible than derived categories of rings, at least by current technology.

Moreover, allowing general triangulated categories as factors of recollements definitely produces counterexamples to a general form of Theorem 5.1, even for very small and natural examples, as the following example shows:

Example 6.1. *This example is taken from [1, Example 5.1], where more detail is given.*

Let A be the Kronecker algebra over an algebraically closed field k . This is a hereditary algebra with two simple modules, whose derived category is equivalent to the category of coherent sheaves on a projective line, by [3]. It has obvious recollements, where the two factors each are equivalent to the derived category of $\text{Mod-}k$, which is clearly derived simple.

However, there is a rather different recollement of the following form:

$$\mathcal{D}(A_t) \quad \begin{array}{c} \xleftarrow{\quad} \\[-1ex] \xrightarrow{\quad} \end{array} \quad \mathcal{D}(A) \quad \begin{array}{c} \xleftarrow{\quad} \\[-1ex] \xrightarrow{\quad} \end{array} \quad \text{Tria } t$$

Here, $A_{\mathbf{t}}$ is a simple artinian ring, thus derived simple, but not Morita equivalent to k . And the triangulated category $\text{Tri}_{\mathbf{t}}$ on the right hand side, generated by the regular modules, that is by the tubes in the Auslander Reiten quiver, can be decomposed further, since there are no maps or extensions between different tubes - thus we can iterate forming recollements infinitely many times, producing an infinite derived composition series.

The terms of this recollement are obtained as follows: We consider the class of indecomposable regular right A -modules \mathbf{t} . By the Auslander-Reiten formula the tilting class

$$\mathbf{t}^\perp = {}^o\mathbf{t}$$

is the torsion class of all *divisible modules*. There exists a tilting module W which generates \mathbf{t}^\perp . The module W can be chosen as the direct sum of a set of representatives of the Prüfer A -modules and the generic A -module G . Moreover, there is an exact sequence

$$0 \rightarrow A \rightarrow W_0 \rightarrow W_1 \rightarrow 0$$

where $W_0 \cong G^d$, and W_1 is a direct sum of Prüfer modules. Then W is equivalent to the tilting module $A_{\mathbf{t}} \oplus A_{\mathbf{t}}/A$, and there is the above recollement, where $A_{\mathbf{t}} \cong \text{End}_A(W_0) \cong (\text{End}_A(G))^{d \times d}$.

Thus, a general version of 5.1 would fail rather dramatically even in this easy situation.

From this discussion we can conclude that the question of validity of a Jordan Hölder theorem has to be restricted to stratifications with all factors being derived categories of rings. We are left with the following problem, which like in the classical situations has a negative answer - of course, some finiteness assumptions are needed in 5.1.

Problem. Given a ring A , do all stratifications of $\mathcal{D}(A)$ by derived module categories of rings have the same finite number of factors, and are these factors the same for all stratifications, up to ordering and up to derived equivalence?

As to be expected, on this level of generality, the problem has a negative answer. The next example is a counterexample; it shows that the number of factors may be infinite.

Example 6.2. Let k be a field, and $A = k^{\mathbb{N}}$ the direct product of countably many copies of k . Then $\mathcal{D}(A)$ has an infinitely long stratification. More precisely, it has a recollement with itself occurring as one factor:

$$\mathcal{D}(A) \quad \begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \\ \longleftrightarrow \end{array} \quad \mathcal{D}(A) \quad \begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \\ \longleftrightarrow \end{array} \quad \mathcal{D}(k)$$

Let $e_1 = (1, 0, 0, \dots) \in A$ be the idempotent supported on the first index. Then $e_1 A$ is finitely generated projective with endomorphism ring k , and the universal localization of A at $e_1 A$ is a homological epimorphism since A is von Neumann regular. By [1, 4.5], $e_1 A$ induces a recollement. The ring on the left hand side is $A/\tau_{e_1 A}(A)$, which is isomorphic to A itself.

More dramatically, uniqueness of factors in a finite stratification can fail and even the length of finite stratifications is not an invariant. Examples have been constructed by Chen and Xi [5].

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